Renormalization group calculation of anomalous exponents for nonlinear diffusion

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We consider the heat equation with nonlinearity, $u_t = u_{xx} + \epsilon f(x,u,u_x,u_{xx})$, where ϵ is a small parameter. Using a renormalization group approach, we calculate that for large space and time, the solution is characterized by $u(x,t) \sim t^{-1/2-\alpha} u^*(xt^{-1/2},1)$, where α is a simple function of the powers of x, u, u_x, and u_{xx} in f. The same approach can be used to calculate the exponent and coefficient for finite time blow-up of equations such as $u_t = u_{xx} + u^r$, where $r > 1$. In both cases the calculations can be performed within the standards of asymptotic analysis.

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I. INTRODUCTION

The renormalization group (RG) methods of Wilson and Kogut [1], and Wilson and Fisher [2] that were very successful in determining critical exponents of statistical mechanics have been broadened in recent years to include a spectrum of problems such as fractals and difference equations (see Creswick, Farach, and Poole [3] for an excellent survey). In particular, applications to nonlinear differential equations have been considered by Glimm, Zhang, and Sharp [4] for chaotic mixing of interfaces, Zhang [5] for random velocity fields, Avallaneda and Majda [6] for stochastics and turbulent transport, and Goldenfeld, Martin, Oono, and Liu [7,8], which we denote (GMOL), in the case of the porous medium equation. An anomalous scaling exponent of the form $u(x,t) \sim t^{-1/2-\alpha}$ is determined using RG methods in GMOL. Bricmont, Kupianen, and Lin [9,0] have also obtained some existence proofs for nonlinear parabolic equations using renormalization group techniques.

In this paper we consider the heat equation with a broad class of nonlinearities and calculate a set of anomalous exponents using methods similar to GMOL. An additional objective is to develop the calculations along the lines of standard applied analysis without relying upon analogies from other physical applications or the introduction of different length scales. Within this framework, the steps necessary for making the calculation mathematically rigorous are evident.

The class of equations we consider is of the form

$$u_t = u_{xx} + f(x,u,u_x,u_{xx}) , \tag{1.1}$$

where f is a polynomial of its arguments and is restricted by dimensional analysis considerations as discussed in Sec. II. We conclude that the long time and large scale evolution of Eq. (1.1) with a thin Gaussian of characteristic length l as an initial condition is governed by

$$u(x,t) \sim t^{-(1/2+\alpha)} u^*(x/t^2,1) , \tag{1.2}$$

where $\alpha = (-1)^{p+1} 1 \times 3 \times 5 \cdots \times (p-1)\epsilon B$ for the nonlinearity $\epsilon B x^m u^n u_x^p$ in f. A somewhat more complicated formula is derived for nonlinear terms involving u_{xx} terms. The nonlinearities we consider are those which arise without the introduction of other dimensional parameters, so that the exponent p characterizes the nonlinearity. As noted in GMOL, standard dimensional analysis cannot be used to calculate the exponent α. In many cases such exponents arise as a result of a limit of vanishing length (or other) scale that is a singular rather than a regular perturbation (as discussed in Barenblatt [11]). In this case the dimensionless small number ϵ appears to provide the correction to the classical exponent.

Equations of the form of (1.1) arise in a broad range of diffusion problems in which the detailed physics is taken into account (see, for example, Ozisik [12], and Gebhart [13]). Prigogine [14], pp. 55–68, discusses the limitations of the linear theory of diffusion and derives a number of the key nonlinearities of the form of (1.1) from basic thermodynamics. From a macroscopic perspective, a basic source of nonlinearities involves inhomogeneities in the diffusion coefficient in the flux (Shewmon [15], p. 6) or variable dependent potentials ([15], p. 25) in Fick's laws. Particular examples involve (i) temperature dependent heat conduction, (ii) compressible fluid flow equations McComb [16], (iii) phase transitions involving alloys [17], (iv) seepage flow in which the permeability is dependent on the absolute value of the flow velocity Muscat [18], (v) magnetic fields with permeability depending upon field strength Jackson [19], Chap. 6, (vi) heat diffusion and phase transition problems in which $(\text{temperature})^{-1}$ dependence is considered [14,20], and many other applications [21].

The methodology presented in this paper is useful not only for the exact calculation of large time profiles, but also in establishing equivalence classes in nonlinearities, since the exponents are determined by a simple formula. This also makes possible additional criteria for deciding on models that agree with experiment.

Our analysis also establishes a close link between the large time asymptotics and the blow-up problems (in which u diverges) by using renormalization group methodology. A classical blow-up problem that has been studied extensively is Eq. (1.1) with f defined as u^r for $r > 1$. Berger and Kohn [22] have used rescaling arguments as part of a numerical scheme to determine the singularity.

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Recently, Bricmont and Kupiainen [10] have provided rigorous proofs of the existence of infinitely many profiles around the blow-up point.

II. ASYMPTOTICS OF THE HEAT EQUATION WITH SMALL NONLINEARITY

Let ϵ be a small dimensionless number. We consider the heat equation with nonlinearity in the form

$$C_p u_{t'} = K\{u_{xx} + \epsilon N[x,u,u_x,u_{xx}]\} , \quad (2.1)$$

where C_p and K are constants (with $D := K/C_p$) and N is a sum of terms of its arguments of the form

$$x^m u^n u_x^p u_{xx}^q , \quad (2.2)$$

where m, n, p, q are integers that satisfy

$$n+p+q=1 , \quad (2.2a)$$

$$p+2q-m=2 , \quad (2.2b)$$

so that (2.1) is dimensionally correct without the introduction of a new time or space scale. We also assume that u is dimensional (e.g., temperature) so that (2.2a) is required. Examples are (i) $u^{-1}u_x^2$, (ii) $x^{-1}u_x$, (iii) $xu^{-2}u_x^3$, (iv) $x^2u^{-2}u_x u_{xx}^2$. Note that (i) arises from the inverse temperature diffusion,

$$(u^{-1})_{t'} = D(u^{-1})_{xx} \text{ or } u_{t'} = D(u_{xx} - 2u_x^2/u) . \quad (2.3)$$

In general, diffusion processes obtained in the manner of (2.3) that do not involve additional physical constants will lead to these types of nonlinearities.

To simplify notation and maintain correspondence with GMOL we define $t := 2Dt'$ (units *length*^{2}) and use (2.1) in the form

$$u_t = \tfrac{1}{2}u_{xx} + \epsilon N[x,u,u_x,u_{xx}] , \quad (2.1')$$

subject to the initial condition

$$u(x,0;l) := g(x;l) := \frac{Q_0}{(2\pi l^2)^{1/2}} \exp\left[\frac{-x^2}{2l^2}\right] , \quad (2.4)$$

where $Q_0 := T_0 Q_1$ with T_0 having temperature units and Q_1 length units. We will be interested in a sharply peaked Gaussian so that l will be small. One of the subtleties in the asymptotics, however, is that the scale of l compared with ϵ must be controlled. For the sake of continuity, we postpone discussion of the error terms until Sec. IV. Using Green's function

$$G(x,t) := (2\pi t)^{-1/2} \exp\left[\frac{-x^2}{2t}\right] , \quad (2.5)$$

and treating the nonlinearity as a source term we can express the solution to (2.1′) as

$$u(x,t) = \int_{-\infty}^{\infty} dy\, G(x-y,t) g(y) + \epsilon \int_0^t ds \int_{-\infty}^{\infty} dy\, G(x-y,t-s) \times N[y,u(y,s),\dots] . \quad (2.6)$$

We solve this using an asymptotic expansion for small ϵ and write the formal sum,

$$u(x,t;\epsilon,l) = u_0(x,t;l) + \epsilon u_1(x,t;l) + \cdots , \quad (2.7)$$

so that l is not yet treated as a small number in comparison with ϵ. Formally solving (2.6) by substituting (2.7) and retaining only $O(1)$ terms leads to the expression

$$u_0(x,t) = \frac{Q_0}{[2\pi(t+l^2)]^{1/2}} \exp\left[\frac{-x^2}{2(t+l^2)}\right] , \quad (2.8)$$

where we have suppressed the parameter l. The derivatives of u_0 are given by

$$\frac{\partial u_0}{\partial x} = \left[\frac{-x}{t+l^2}\right] u_0, \quad \frac{\partial^2 u_0}{\partial x^2} = (t+l^2)^{-1}\left[\frac{x^2}{t+l^2} - 1\right] u_0 , \quad (2.9)$$

so that one has the relation

$$\frac{\partial^2 u_0}{\partial x^2} = -(t+l^2)^{-1}\left[u_0 + x\frac{\partial u_0}{\partial x}\right] . \quad (2.10)$$

The last identity will establish a simple relation between the effects of nonlinearities involving the second order derivatives and the lower order in terms of the anomalous exponents. We proceed by using u_0 in the expression (2.6) to generate the next term of (2.7), namely, u_1. Since N consists of a linear sum of the form (2.2), it suffices to consider the nonlinear term $x^m u^n u_x^p u_{xx}^q$ subject to (2.2). For convenience, we consider the case $q=0$ first, so that the nonlinearity will be completely specified by p, as $n=1-p$ and $m=p-2$. We then have

$$u_1(x,t;l) = \int_0^t ds \int_{-\infty}^{\infty} dy \frac{(t-s)^{-1/2}}{(2\pi)^{1/2}} \exp\left[\frac{-(x-y)^2}{2(t-s)}\right] \frac{Q_0}{(2\pi)^{1/2}} y^m (s+l^2)^{-1/2} \frac{(-y)^p}{(s+l^2)^p} \exp\left[\frac{-y^2}{2(s+l^2)}\right]$$
$$\cong \frac{Q_0}{2\pi} t^{-1/2} e^{-x^2/2t} \int_0^t ds (s+l^2)^{-p-1/2} (-1)^p \int_{-\infty}^{\infty} dy\, y^{p+m} \exp\left[\frac{-y^2}{2(s+l^2)}\right]$$
$$+ (\text{terms that are smaller in } l) . \quad (2.11)$$

The approximations involve replacing $t-s$ by s, and $x-y$ by x to obtain the $t^{-1/2}e^{-x^2/2t}$ term. The justification, which is discussed in detail in Sec. IV, is based on Laplace's method for integrals (Erdelyi [23], p. 36), since the main contribution to the integral must arise from the regions near $y=0$ and $s=0$ for small l. Note that the entire term is of the same order in ϵ so that the smaller terms in l cannot cancel the higher order. Let I_1 denote the integrand of the s integral and use $w:=y/(s+l^2)^{1/2}$ to obtain (for $p\geq 1$)

$$\begin{aligned} I_1&=(s+l^2)^{-1}(-1)^p\int_{-\infty}^{\infty} w^{2(p-1)}e^{-w^2/2}dw \\ &=\frac{(-1)^p(2\pi)^{1/2}}{(s+l^2)}\{1\times 3\times\cdots\times|2(p-1)-1|\} . \end{aligned} \tag{2.12}$$

Combining this with (2.8) one has to leading order in ϵ and to leading order in l within $O(\epsilon)$ the solution

$$u(x,t;\epsilon,l)=\frac{Q_0t^{-1/2}}{(2\pi)^{1/2}}e^{-x^2/2t}\{1+\epsilon(-1)^p 1\times 3\times\cdots\times|2p-3|\ln(t/l^2)\} , \tag{2.13}$$

for $p\geq 1$ and $q:=0$.

Nonlinearities involving u_{xx}. The nonlinearity $x^m u^n u_x^p u_{xx}^q$ (with $q>0$) leads to

$$\begin{aligned} u_1(x,t;l)&\cong\frac{Q_0}{2\pi}t^{-1/2}e^{-x^2/2t}\int_0^t ds\, I_1(s,y;l) , \\ I_1&:=(s+l^2)^{-1}(-1)^p\int_{-\infty}^{\infty} w^{p+m}(w^2-1)^q e^{-w^2/2}dw . \end{aligned} \tag{2.14}$$

Using the binomial theorem to evaluate I_1, one obtains for

$$N[x,u,u_x,u_{xx}]=\sum_{m,n,p,q} B(m,n,p,q)x^m u^n u_x^p u_{xx}^q$$

the result

$$\begin{aligned} u(x,t;\epsilon,l)=&\frac{Q_0t^{-1/2}}{(2\pi)^{1/2}}e^{-x^2/2t} \\ &\times\left\{1+\epsilon\sum_{m,n,p,q}B(m,n,p,q)\sum_{j=0}^{q}(-1)^{j+p}1\times 3\times\cdots\times|2p+4q-2j-3|\ln(t/l^2)\right\} . \end{aligned} \tag{2.15}$$

In particular, for the special case in which $m=n=p=0$ and $q=1$ one obtains

$$\sum_{j=0}^{1}(-1)^j 1\times 3\times\cdots\times|4-2j-3|\ln(t/l^2)=0 ,$$

indicating as expected that the addition of this linear term u_{xx} does not make a contribution and will not change the exponent.

Porous medium equation. The porous medium equation considered by GMOL can be written as

$$u_t-\tfrac{1}{2}u_{xx}=\frac{\epsilon}{2}H(-u_{xx})u_{xx} , \tag{2.16}$$

where $H(z):=1$, if $z>0$ and vanishes otherwise. Hence,

$$H(-u_{0yy})=H\left[1-\frac{y^2}{s+l^2}\right] ,$$

and the integral $\int_{-\infty}^{\infty}(w^2-1)e^{-w^2/2}dw=0$ is truncated to $\int_{-1}^{1}(w^2-1)e^{-w^2/2}dw\neq 0$ so a nontrivial contribution to u_1 via (2.14) is possible. Thus it appears that all but a small fraction of nonlinearities similar to (2.16) will result in a nonzero contribution that will lead to an anomalous exponent. In other words the integral

$$\int_{-\infty}^{\infty}F((w^2-1)e^{-w^2/2})(w^2-1)e^{-w^2/2}dw=0 \tag{2.17}$$

will vanish for $F(x):=1$, the linear case, and for a set of functions that represent a particular symmetry that weights the function equally on either side of $w=1$ with respect to the particular symmetry that originates from the second derivative of the Gaussian.

III. THE RENORMALIZATION GROUP FROM AN ANALYSIS PERSPECTIVE

Given an asymptotic relation such as (2.15) one can calculate the anomalous exponent explicitly and obtain the precise similarity solution for large time and space. The arguments are within the context of formal applied analysis without reference to numerical procedures or physical analogies, and are thus in a form that can provide a basis for attempting rigorous proofs and generalizations to a wide variety of nonlinear differential equations. The methodology is close in spirit to those of Goldenfeld, Martin, and Oono [8] and Creswick, Farach, and Poole [3]. For the problem under consideration, the result can be stated as follows (using true dimensions):

Proposition 3.1. Suppose u can be expressed as

$$u(x,t';\epsilon,l)=\frac{T_0\left[\dfrac{t'}{Q_1^2/D}\right]^{-1/2}}{2\pi^{1/2}}e^{-x^2/(4Dt')}\times\{1+\epsilon A\ln(2Dt'/l^2)\}\ , \tag{3.1}$$

where A does not depend upon x, t', ϵ or l, and $\epsilon A\ln(2Dt'/l^2)\ll 1$. Then, to leading order in ϵ, u can be written as

$$u(x,t';\epsilon,l)=\left[\frac{t'}{Q_1^2/D}\right]^{-(1/2)+\epsilon A}\times u^*\left[\frac{x}{(Dt'/Q_1^2)^{1/2}},\frac{Q_1^2}{D}\right]\ , \tag{3.2}$$

so that the anomalous exponent is given by ϵA. The fixed point function u^* is given by

$$u^*(\zeta,\tau_1)=\frac{T_0}{2\pi^{1/2}}\exp\left\{-\frac{\zeta^2}{4D\tau_1}\right\}\left\{1+\epsilon A\ln\left[\frac{2D}{l^2}\tau_1\right]\right\}\ . \tag{3.3}$$

Verification. The derivation is divided into five stages that can be implemented more generally on other problems.

Stage 1. One needs to obtain an identity [up through $O(\epsilon)$] of the form

$$u(b^{\phi}x,bt')=Z(b)u(x,t')\ , \tag{3.4}$$

that is, valid for a particular choice of Z and ϕ and all $b>1$. For the problem under consideration, clearly the exponential term in (3.1) forces $\phi=1/2$. Rewriting (3.1) to $O(\epsilon)$ one has

$$u(b^{1/2}x,bt')=\frac{T_0}{(2\pi)^{1/2}}\left[\frac{2t'}{Q_1^2/D}\right]^{-1/2}b^{-1/2}e^{-x^2/4Dt'}\times\{1+\epsilon A\ln(t/l^2)\}\{1+A\ln b\}\ , \tag{3.5}$$

so that (3.4) is satisfied with $\phi=1/2$ and

$$Z(b):=b^{-1/2}(1+\epsilon A\ln b)\ . \tag{3.6}$$

Note that Z does not depend upon l.

Following Creswick, Farach, and Poole [3] we define the operator

$$R_{b,\phi}u(x,t'):=\frac{1}{Z(b)}u(b^{1/2}x,bt')\ . \tag{3.7}$$

Stage 2. By iteration we have [again suppressing ϵ and l and ignoring $O(\epsilon^2)$ terms],

$$u(b^{k/2}x,bt')=Z(b)^k u(x,t')\ . \tag{3.8}$$

A fixed point of this iteration will exist only if

$$u^*(x,t'):=\lim_{k\to\infty}Z(b)^{-k}u(b^{k/2}x,b^k t') \tag{3.9}$$

is well defined. We assume the existence of a fixed point in this formal derivation and rewrite (3.9) for large but finite k as

$$u(b^{k/2}x,b^k t')\cong Z(b)^k u^*(x,t')\ . \tag{3.10}$$

Note that $b>1$ was necessary for considering large time and space, and in fact for the assumption of approximate self-similarity that underlies the existence of the fixed point u^*. We rewrite the last equation by defining

$$\bar{x}:=b^{k/2}x,\quad \bar{t}:=b^k t'\ , \tag{3.11}$$

so that one has (for large k)

$$u(\bar{x},\bar{t})\cong Z(b)^k u^*(\bar{x}b^{-k/2},\bar{t}b^{-k})\ . \tag{3.12}$$

This means that for any large $\bar{t}$ we can determine the u profile by setting $b^k:=\bar{t}/(Q_1^2/D)$, so that the second argument remains unchanged as we examine different values of $\bar{t}$, and we can then write (3.12) as

$$u(\bar{x},\bar{t})\cong\left\{Z\left[\left[\frac{\bar{t}}{Q_1^2/D}\right]^{1/k}\right]\right\}^k u^*\left[\frac{\bar{x}}{(D\bar{t}/Q_1^2)^{1/2}},\frac{Q_1^2}{D}\right]\ . \tag{3.13}$$

Note that if we chose to ignore the units we would set the second argument of u^* at unity.

Stage 3. The scaling exponent will be determined by the limit

$$\lim_{k\to\infty}\left\{Z\left[\left[\frac{\bar{t}}{Q_1^2/D}\right]^{1/k}\right]\right\}^k\ ,$$

if it exists. To calculate this we let $t_1:=D\bar{t}/Q_1^2$ and substitute directly into (3.6) and utilize the asymptotic expansion

$$\left[1+\frac{\epsilon A}{k}\ln t_1\right]^k\cong t_1^{\epsilon A}\ , \tag{3.14}$$

to obtain

$$[Z(t_1^{1/k})]^k=t_1^{-1/2}\left\{1+\frac{\epsilon A}{k}\ln t_1\right\}^k\cong t_1^{(-1/2)+\epsilon A}\ ,$$

yielding the result

$$\lim_{k\to\infty}\left\{Z\left[\frac{D\bar{t}}{Q_1^2}\right]^{1/k}\right\}^k=(D\bar{t}/Q_1^2)^{-(1/2)+\epsilon A}\ . \tag{3.15}$$

Stage 4. Using (3.15) in (3.13), and dropping the overbar since (3.13) is valid for arbitrary large $\bar{t}$, one obtains

$$u(x,t')=(Dt'/Q_1^2)^{-(1/2)+\epsilon A}u^*\left[\frac{x}{(Dt'/Q_1^2)^{1/2}},\frac{Q_1^2}{D}\right]\ , \tag{3.16}$$

so that the anomalous exponent or "dimension" is $\alpha=-\epsilon A$.

Stage 5. Explicit evaluation of u^* is possible by writing (3.1) as

$$u(x,t';\epsilon,l)=\frac{T_0\left[\frac{t'}{Q_1^2/D}\right]^{-1/2}}{2\pi^{1/2}}e^{-x^2/(4Dt')}\times\{1+\epsilon A\ln(Dt'/Q_1^2)\}\times\left\{1+\epsilon A\ln\left[\frac{2Q_1^2}{l^2}\right]\right\},$$

and utilizing (3.14) again to obtain

$$u(x,t';\epsilon,l)=\frac{T_0}{2\pi^{1/2}}\left[\frac{Dt'}{Q_1^2}\right]^{-(1/2)+\epsilon A}\times e^{-x^2/(4Dt')}\left\{1+\epsilon A\ln\left[\frac{2Q_1^2}{l^2}\right]\right\}. \quad (3.17)$$

Comparison of (3.17) with (3.16) leads to the evaluation of u^* as

$$u^*\left[\frac{x}{(Dt'/Q_1^2)^{1/2}},\frac{Q_1^2}{D}\right]=\frac{T_0}{2\pi^{1/2}}\exp\left\{-\frac{\left[\frac{x}{(Dt'/Q_1^2)^{1/2}}\right]^2}{4D(Q_1^2/D)}\right\}\left\{1+\epsilon A\ln\left[\frac{2Q_1^2}{D}\frac{D}{l^2}\right]\right\}, \quad (3.18)$$

which is (3.3) so Proposition 3.1 has been verified.

The results of Sec. II then imply the following result.

Proposition 3.2. The general nonlinearity $N[x,u,u_x,u_{xx}]$ defined in (2.15) has the anomalous exponent

$$\alpha=-\epsilon\sum_{m,n,p,q}B(m,n,p,q)\sum_{j=0}^{q}(-1)^{j+p}1\times3\times\cdots\times|2p+4q-2j-3| . \quad (3.19)$$

While the calculation has used the idea of small ϵ, the expectation of universality would imply that the exponent varies continuously as ϵ is made larger provided a singularity does not occur due to the nonlinearity.

IV. ERROR TERMS

The analysis of Sec. II involves approximations along the lines of (i) small ϵ, (ii) small l, (iii) large t and x. The RG calculation involves the additional approximation involved in the fixed point in that (iv) large k is used to approximate ∞. The interactions between these approximations present a number of subtleties that require clarification. While the derivation reduces the calculation of exponents to simple arithmetic, the actual existence of the fixed point u^* is subtle and its existence is contingent upon the boundedness of the nonlinear terms that are small in terms of the formal analysis.

We now analyze the integrals of Sec. III in terms of the error involved in the approximations beginning with the first expression of (2.11) which we write as

$$u_1(x,t)=\frac{(-1)^pQ_0}{2\pi}\int_0^t ds\frac{(t-s)^{-1/2}}{(s+l^2)^{p+1/2}}J_1(t,s) , \quad (4.1)$$

$$J_1(t,s):=\int_{-\infty}^{\infty}dy\exp\left[\frac{-(x-y)^2}{2(t-s)}\right]\times\exp\left[\frac{-y^2}{2(s+l^2)}\right]y^{m+p} . \quad (4.2)$$

We will divide both integrals (ds and dy) into two parts, namely, the small s part, i.e., $s\le s_0$, and small y part, $|y|\le y_0$, and the remainder terms in the integrals. For the small s and y parts it is useful to write the first exponential of (4.2) as a double Taylor series

$$\exp\left[\frac{-(x-y)^2}{2(t-s)}\right]=e^{-x^2/(2t)}\sum_{i,j}A_iB_j(y/x)^i(s/t)^j .$$

These are convergent series for $|y/x|<1$ and $|s/t|<1$ in which the first term ($i=0$, $j=0$) is 1. Using this expression in (4.2) we write

$$J_1(t,s)=e^{-x^2/(2t)}\sum_{i,j}A_iB_j(s/t)^j\times\int_{-\infty}^{\infty}dy(y/x)^i\exp\left[\frac{-y^2}{2(s+l^2)}\right]y^{m+p} . \quad (4.3)$$

Now define $J_{11}:=\int_{-y_0}^{y_0}\cdots$ as the small y part of the J_1 integral. The difference between the two is then

$$\tfrac{1}{2}|J_1-J_{11}|\le\int_{y_0}^{\infty}dy\exp\left[\frac{-y^2}{2(s+l^2)}\right]y^{m+p} , \quad (4.4)$$

for $t>s$ as a consequence of the inequality

$$\exp\left[\frac{-(x-y)^2}{2(t-s)}\right]\le1 .$$

Using the substitution $w:=y/(s+l^2)^{1/2}$ again, we will obtain a bound on the difference (4.4) by defining

$$G(z):=\int_z^\infty dw\, w^{m+p}e^{-w^2/2}$$
$$=\int_z^\infty dw\, w^{m+p-1}e^{-w^2/4}we^{-w^2/4} . \tag{4.5}$$

The bound, $w^{m+p-1}e^{-w^2/4}\le C(m,p)$, where $C(m,p)$ is a constant depending only on m and p, then yields the result,

$$G(z)\le 2C(m,p)e^{-z^2/4} . \tag{4.6}$$

Using this in (4.4), one obtains the bound

$$\tfrac{1}{2}|J_1-J_{11}|\le 2C(m,p)(s+l^2)^{-(m+p+1)/2}$$
$$\times e^{-(s+l^2)^{-1}y_0^2/4} . \tag{4.7}$$

Note that $|J_1-J_{11}|$ is a function of t and s, and depends upon the parameters l and y_0. We choose y_0 and s_0 as suitable functions of l so that the bound (4.7) is adequate to show that J_{11} differs from J_1 only by terms that are exponentially small in l. One way of accomplishing this is by choosing

$$y_0:=l^{1/8},\quad s_0:=l^{1/2} . \tag{4.8}$$

(An alternative employing original units would be to use a large constant c_1 and set $y_0:=c_1 l$ and $s_0:=c_1 l$.) Then (4.7) implies

$$\tfrac{1}{2}|J_1-J_{11}|(t,s)\le(s+l^2)^P e^{-l^{-1/4}} \quad \text{if } s\le s_0 , \tag{4.9}$$

for some integer P. Combining the bound (4.9) with the expression (4.3), we obtain, by splitting the s integral into two parts, the expression

$$u_1(x,t)=\frac{(-1)^pQ_0}{2\pi}\int_0^{s_0}ds\frac{(t-s)^{-1/2}}{(s+l^2)^{p+1/2}}J_{11}(t,s)$$
$$+\frac{(-1)^pQ_0}{2\pi}\int_{s_0}^\infty ds\frac{(t-s)^{-1/2}}{(s+l^2)^{p+1/2}}J_{11}(t,s)$$
$$+O(e^{-l^{-1/4}}) . \tag{4.10}$$

Focusing on the first integral, we note that $t\gg s\gg s_0$ and we can expand part of the integrand in a Taylor series as

$$(t-s)^{-1/2}=t^{-1/2}\sum_k C_k(s/t)^k , \tag{4.11}$$

where the coefficient of the first term in the series is again 1. Now u_1 can be rewritten in the form

$$u_1(x,t)=\frac{(-1)^pQ_0t^{-1/2}e^{-x^2/(2t)}}{2\pi}\sum_{i,j,k}A_iB_jC_k\int_0^{s_0}ds\left[\frac{s}{t}\right]^{j+k}(s+l^2)^{-p-1/2}$$
$$\times\int_{-y_0}^{y_0}dy\left[\frac{y}{x}\right]^i y^{m+p}\exp\left[\frac{-y^2}{2(s+l^2)}\right]+O(te^{-l^{-1/4}})+(\cdots)\int_{s_0}^t\cdots . \tag{4.12}$$

Next, we need to extend the dy integral to $(-\infty,\infty)$. Using the same estimate as in (4.6) for $G(z)$, we see that the integral can be extended to the entire real line with an exponentially small error in l.

The term $\int_{s_0}^t$ is not negligible but contains terms that are less singular than the corresponding terms in the $\int_0^{s_0}$ integral. Consequently the leading order term in $\int_0^{s_0}$ cannot be canceled by the upper part of the integral.

Finally, the issue of the double expansion in ϵ and l can be addressed by expressing one in terms of the other and obtaining an expansion in terms of a single parameter. Since ϵ is the magnitude of the nonlinearity and l a length scale associated with the initial condition, it is reasonable to choose l as a function of ϵ. For proper dimensional consistency, we would need to specify l/t_0^2 where t_0 is the minimum time of interest that we used above. Suppressing the t_0 factor, one can write the basic expansion as

$$u=u_0+\epsilon u_1+\cdots$$
$$=u_0+\epsilon\{\ln(t/l^2)u_{10}+(l^2/t)\ln(t/l^2)u_{11}+\cdots\} , \tag{4.13}$$

where u_{10} and u_{11} represent the coefficients in the l expansion of u_1. To obtain a single expansion, we need to guarantee that

$$\delta:=\epsilon\ln(t/l^2)\ll 1 ,$$
$$\epsilon(l^2/t)\ln(t/l^2)\ll O(\delta) . \tag{4.14}$$

Choosing $l:=\epsilon t_0^{1/2}$ ensures these two conditions so that the asymptotic series (4.13) has the form

$$u=u_0+\epsilon u_1+\cdots$$
$$=u_0+\epsilon\{[2\ln\epsilon+\ln(t_0/t)]u_{10}$$
$$+\epsilon(t_0/t)[2\ln\epsilon+\ln(t_0/t)]u_{11}+\cdots\} . \tag{4.15}$$

One can regard this approach (along the lines suggested in Sec. II) as first taking ϵ to be a small parameter and then reducing l to the size of ϵ.

The anomalous exponent can be obtained directly from (4.15). Our analysis of the approximations thus shows that the RG procedure can be reduced to standard asymptotic expansion in ϵ.

V. RG CALCULATION OF BLOWUP

The RG analysis of Sec. III extends easily to blowup of solutions to differential equations. In particular, we consider the equation

$$u_t = u_{xx} + u^r , \tag{5.1}$$

with $r > 1$. Note that a time scale τ is needed as a coefficient of u^r in (5.1) (provided that u is dimensionless, e.g., u =concentration). If u is dimensional, e.g., u =temperature, then the coefficient τ^{-1} is replaced by $\tau^{-1}T_0^{-r+1}$. We suppress these units at this stage since the issue is essentially the same as in Sec. III. One can regard (5.1) as the natural units in which time is measured by τ and space by $(D\tau)^{1/2}$.

Berger and Kohn [22], and references therein, consider the interval $-1 < x < 1$ with Dirichlet boundary conditions

$$u(-1,t') = u(1,t') = 0 , \tag{5.2}$$

and initial data, $\phi(x)$, such that

$$\phi > 0, \quad \phi(x) = \phi(-x), \quad x\phi'(x) < 0 \quad \text{for } x \neq 0 , \tag{5.3}$$

for which the solution to (5.1), (5.2) is known to be positive, symmetric, and decreasing in $|x|$. Furthermore, it is known (Giga and Kohn [24], Galaktionov and Posashkov [25]; see also references within) that the solution to (5.1)–(5.3) exhibits a divergence at some time $t_0 > 0$ so that

$$u(x,t) \sim \left[\frac{t_0 - t}{r+1} \right]^{-1/(r+1)} \tag{5.4}$$

is the leading order (in $|t_0 - t|^{-1}$) term in the solution. Our analysis here is local and applies to any set of initial and boundary conditions for which the solution has similar qualitative behavior.

We will derive (5.4) using the methods of Sec. III so that the coefficient, in addition to the exponent of the leading term, is obtained through a simple calculation. The arguments are close to those of Berger and Kohn [22], who based a numerical scheme on the identity (5.5) below in order to calculate the singularity.

Geometrically, it is easy to see why the methods of Sec. III should apply. In the asymptotic decay problems, the fixed point of the RG methods amounted to a near self-similarity as the solution decayed to zero as t approached ∞. For the blow-up problems there is an analogous situation in that the solution is nearly self-similar as t approaches ∞. To put it more simply, the picture in the blow-up problems looks like the decay problems upon rotating the graph counterclockwise by $\pi/2$.

To define the proper RG operator $R_{b,\phi}$ one observes as before that only $\phi = \frac{1}{2}$ will be possible since the leading space derivative is second order, so that scaling time by a factor b forces a scaling of space by $b^{1/2}$. Substitution of $u(b^{1/2}x, bt)$ into the differential Eq. (5.1) leads immediately to the conclusion that u itself must be scaled by a factor of $b^{1/(1+r)}$, so that if $u(x, t-t_0)$ solves (5.1) then so does

$$R_{b,1/2}u(x,t-t_0) = u_b(x,t-t_0) = b^{1/(1+r)}u(b^{1/2}x, b(t-t_0)) . \tag{5.5}$$

Defining $Z(b) := b^{1/(1+r)}$ one can write the transformation in the standard form as in Sec. III. A repeated application of this transformation with $b < 1$ will move the solution closer to the singularity and closer to self-similarity. In the limit one may expect the existence of a fixed point function u^* satisfying

$$u^*(x,t-t_0) = \lim_{k\to\infty} b^{k/(1+r)}u(b^{k/2}x, b^k(t-t_0)) . \tag{5.6}$$

Letting $\bar{x} := b^{k/2}x$ and $\bar{t} := b^k(t-t_0)$ and writing (5.6) as an approximation for large k, one has (to leading order in k^{-1})

$$u(\bar{x},\bar{t}) = b^{-k/(r+1)}u^*(b^{-k/2}\bar{x}, b^{-k}\bar{t}) . \tag{5.7}$$

For any (small) $\bar{t}$ we can evaluate $u(\bar{x},\bar{t})$ by setting $b^k = -\bar{t}$ so that

$$u(\bar{x},\bar{t}) = (-\bar{t})^{-1/(r+1)}u^*(\bar{x}/\sqrt{-\bar{t}}, 1) , \tag{5.8}$$

for $\bar{t}$ near 0 and remains valid as an approximate solution to (5.1) when $\bar{t}$ is replaced by $t - t_0$. Hence the exponent $-1/(r+1)$ is thereby determined. One can proceed to find the coefficient by substituting this expression into the original differential equation. Upon defining

$$f(\xi) := u^*(\bar{x}/\sqrt{-\bar{t}}, 1), \quad \xi := x/\sqrt{t_0 - t} , \tag{5.9}$$

one can write the differential equation to leading order in $|t_0 - t|^{-1}$ as

$$f''(\xi) + \frac{\xi}{2}f'(\xi) - \frac{1}{1+r}f(\xi) + f(\xi)^r = 0 , \tag{5.10}$$

with solution

$$f(\xi) = (1+r) - 1/(r-1) . \tag{5.11}$$

Substitution of (5.11) into (5.8) implies the formula (5.4), which in more precise form can be written as

$$\lim_{t \nearrow t_0} \frac{u(x,t)}{\left[\frac{t_0 - t}{r+1} \right]^{-1/(r+1)}} = 1 . \tag{5.12}$$

Remark 5.1. Similar conclusions about the nature of blow-up can be drawn from guessing the form

$$u(x,t) = t^{-s}f\{x(t_0 - t)^{-1/2}\} , \tag{5.13}$$

since substitution into the differential equation shows that a solution can be attained with the exponent and functional form of f given by (5.4).

VI. CONCLUSION

We have shown that renormalization methods can be used to calculate the anomalous exponent of the heat equation with a class of nonlinearities. Furthermore, this can be done within a systematic applied mathematical setting of asymptotic analysis. The axiomatic set of steps involved in the calculation thus has the potential for complete rigor and does not rely upon physical analogy with other phenomena.

The renormalization group within this setting becomes a simple computational tool that can be generalized to other classes of partial differential equations whenever there is basic solution upon which perturbations (e.g., nonlinearities of order ϵ) can be calculated using the usual asymptotic analysis procedures.

The nonlinearities we have considered include the inverse temperature heat equation, $(T^{-1})_t=(T^{-1})_{xx}$, as a special case. This equation along with source terms has been used in some phase transition problems. For temperatures that are far from absolute zero, the difference between this equation and the ordinary heat equation amounts to a nonlinear source term as noted in Eq. (2.3). The fact that the two equations differ in the large time exponent gives a clear criterion for the use of each in modeling. In fact, this procedure of calculating the large time behavior using these RG methods can be used in conjunction with experiment and statistical mechanics calculations in order to decide on the appropriate equations in a particular problem.

Our analysis also unifies the methodology involved in blowup problems with those of large time decay. The key first step in both problems is to obtain a transformation that leads to a fixed point, indicating that the solution tends asymptotically to a self-similar graph. In the case of the large time decay, the solution approaches self-similarity as (u,t) tends to $(0,\infty)$. In the case of blowup the situation is identical except for a $\pi/2$ rotation of the axes so that self-similarity is approached as (u,t) tends to (∞,t_0). The second step in both cases involves extracting functional relationships based on the existence of a fixed point as the transformation is applied a large number, k, times. Using a large but still finite k, one can obtain the unique functional form that is compatible with the fixed point. This procedure is philosphically similar to using repeated rescaling of the numerical grid but allows a direct and simple calculation of the exponent as well as the coefficient of the singularity. Numerical computation can be avoided in the blowup problems just as appeal to physical analogy can be avoided in the asymptotic decay problems, as the underlying mathematical structure is in fact quite simple.

Throughout this analysis ϵ has been a small parameter. It has been noted in [7,8] that the results could be continued for large ϵ in the spirit of the Wilson-Fisher ϵ expansion. The formalism of Sec. III may be used in conjunction with analytic continuation methods to prove the validity of the expansion beyond the usual rigorous asymptotics. In most cases asymptotic calculations are made rigorous within an arbitrarily small neighborhood $\epsilon\in(0,\epsilon_0)$. The continuation methods may be useful in extending the arbitrarily small ϵ_0 to a known finite number.

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